

## Multiscaling

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**Abstract.** We introduce the unbiased way statisticians look at the 2-point correlation function and study its relation to multifractal analysis. We apply this method to a simulation of the distribution of galaxy clusters in order to check the dependence of the correlation dimension on the cluster richness.

### 1. Introduction

The statistical description of the galaxy clustering is usually based on the two-point correlation function  $\xi(r)$ . This function is, following the terminology used by statisticians working in point field statistics, a second-order characteristic of the point process (Diggle 1993; Stoyan & Stoyan 1994). The first-order characteristic is just the intensity measure  $\lambda(\vec{r})$  (Martínez et al. 1993). Assuming the Cosmological principle, we accept that galaxies in large volumes represent a stationary and isotropic point process, having therefore constant intensity equal to the number density of galaxies per unit volume, denoted by  $n$ .

### 2. The $K$ -function and the correlation dimension

Among all the second-order characteristics of a point-process the most commonly used is the Ripley's  $K$ -function (Ripley 1981) which is easily related to the mean number of galaxies lying in balls of radius  $r$  centred in an arbitrary galaxy,  $\langle N \rangle_r$  (Peebles 1980), or the correlation integral  $C(r)$  (Martínez et al 1995). For a three-dimensional process we have that:

$$\langle N \rangle_r = nK(r) = C(r) = \int_0^r 4\pi n s^2 (1 + \xi(s)) ds \quad (1)$$

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It is obvious that for a homogeneous Poisson process the  $K$ -function is

$$K(r) = \frac{4\pi}{3}r^3 \quad (2)$$

It is well known the power-law behaviour of  $\xi(r)$  for the galaxy distribution,  $\xi(r) = (r/r_0)^{-\gamma}$  with  $\gamma \simeq 1.8$  and  $r_0 \simeq 5 h^{-1}$  Mpc (Davis & Peebles 1983). In the strong clustering regime where  $\xi(r) \gg 1$ , the previous behaviour is translated to a power-law behaviour in the Ripley's function:  $K(r) \propto r^{D_2}$  with  $D_2 \simeq 3 - \gamma$ . The exponent  $D_2$  is known as the correlation dimension. We have argued that this value itself is an important measure of the clustering (Martínez et al. 1995). Several authors have successfully fitted a power-law to the cluster-cluster correlation function. Some of them claimed to have obtained a value of 1.8 for  $\gamma$  (Bahcall & Soneira 1983), but many others have not. Recent calculations give no hope for a common value of the exponent  $\gamma$  for clusters and galaxies (Postman et al. 1992). For example Dalton et al. (1994) found that  $\gamma \simeq 2$  for the APM clusters. The correlation length found for clusters is larger than the one for galaxies,  $r_{0,c} \simeq 15 - 30 h^{-1}$  Mpc. However a strong controversy has arisen about the value of  $r_{0,c}$  (Bahcall & Soneira 1983; Bahcall et al. 1986, Sutherland 1988; Sutherland & Efstathiou, 1991). In this context one has to be cautious in interpreting the power-law behaviour of  $\xi(r)$  for clusters. Moreover, in the range of scales where  $\xi(r)$  is of the order of unity it is obvious that  $K(r)$  and  $\xi(r)$  cannot simultaneously display power-law behaviour. This could be the case for the distribution of clusters, which in the range  $[10, 50] h^{-1}$  Mpc present values less than 5 for  $\xi(r)$ . We have seen that probably  $K(r)$  fits better a power-law in this range than  $\xi(r)$  does and the exponent  $D_2$  is different for galaxies and for clusters (Martínez et al. 1995).

### 3. Multiscaling

This variation of the relevant exponents with the the kind of object, galaxies of different morphological type or clusters with different richness, might be considered within the framework of the multiscaling hypothesis (Jensen et al. 1991, Paladin et al. 1992). If all these astrophysical objects are interpreted as the consequence of applying different mass thresholds to a continuous density field, the multiscaling approach provides us with a theoretical way to understand the variations in the clustering properties which turn out in variations of the exponent  $D_2$ . The higher the density threshold the stronger the clustering and therefore the smaller the value of the corresponding  $D_2$ . We have tested this approach on several cluster and galaxy simulated samples and we conclude that the multiscaling approach is rather useful for the description of the large scale clustering in the Universe. In fact the dependence of the clustering properties of clusters with different richness is a natural example to illustrate how multiscaling works (Paredes et al. 1995). We shall see here that the use of the function  $K(r)$  with appropriate unbiased estimators is the best tool for the study of the multiscaling distribution in models of clusters of galaxies and at the same time gives us information about the scale where the transition to homogeneity appears.

#### 4. Estimators

The estimators of a statistical quantity as  $K(r)$  need to have several good properties (Stoyan & Stoyan 1994), but obviously the most important one is that the estimator must be unbiased, in the sense that the mean value of the estimates when applied to different samples of a given process has to be equal to the true value of the statistical quantity. At large distances, the unbiasedness of  $K(r)$  is mostly related with the edge-corrections of the estimators. Here we shall use the estimator proposed by Ripley (1976) and Baddeley et al. (1993). For  $N$  galaxies at positions  $\{X_i\}_{i=1}^N$  distributed in a volume  $V$ , it reads:

$$\hat{K}(r) = \frac{V}{N^2} \sum_{i=1}^N \sum_{\substack{j=1 \\ j \neq i}}^N \frac{\theta(r - |\vec{X}_i - \vec{X}_j|)}{w_{ij}}, \quad (3)$$

$\theta$  being the step function, which is 0 for negative values of its argument and 1 for positive ones. The weight  $w_{ij}$  is the proportion of the area of the sphere, centred at  $X_i$  and passing through  $X_j$ , which lies inside the volume  $V$ . In the case of a cubic sample such as we consider, Baddeley et al. (1993) give a (complicated) analytic expression for it that we can use. The sum in Eq. (3) is an unbiased estimator for  $n^2 V K(r)$  (Ripley 1976, 1981; Diggle 1983). For more details see also Martínez & Pons (1996).

#### 5. Results in a model for the distribution of clusters

The fractal behaviour of the distribution of galaxy clusters has a breakdown at some given distance (Borgani et al. 1994), in which the tendency to homogeneity appears in an evident way. Here we shall use a numerical model for the distribution of rich clusters. In this model, clusters are identified as the peaks in the evolved density field of Zel'dovich simulations with  $\Omega = 1$ , where 30% of this critical density is provided by massive neutrinos and the rest by cold particles (Klypin et al. 1993, Borgani et al. 1995). Applying different density thresholds to the simulation we obtain samples resembling the samples of rich galaxy clusters. The higher the threshold the richer the corresponding clusters and the larger the mean separation between them. Using the mean particle separation  $d = n^{-1/3}$  as the control parameter we have extracted 5 samples from one realization of the model. In Figure 1, we show these samples drawn from a simulation in a cube of side  $320 h^{-1}$  Mpc.

In Figure 2 we show the  $K$ -function for each of the previous samples, together with the variation of the correlation dimension  $D_2$  with the scale  $r$ . This latter value has been calculated as the local slope of the log-log plot  $K(r)$  vs.  $r$  in a range of scales of  $10 h^{-1}$  Mpc width centred at each value of  $r$ . We can see that for scales in the range  $[10, 40] h^{-1}$  Mpc the function  $K(r)$  follows reasonably well a power law with exponent  $D_2 < 3$ . However the exponent depends on the density in the way prescribed by the multiscaling hypothesis: for higher density threshold (or equivalently larger interparticle distances), smaller values of  $D_2$  are found, indicating stronger clustering. For scales larger than  $40 h^{-1}$  Mpc, a clear tendency to homogeneity (represented by the dotted line in the

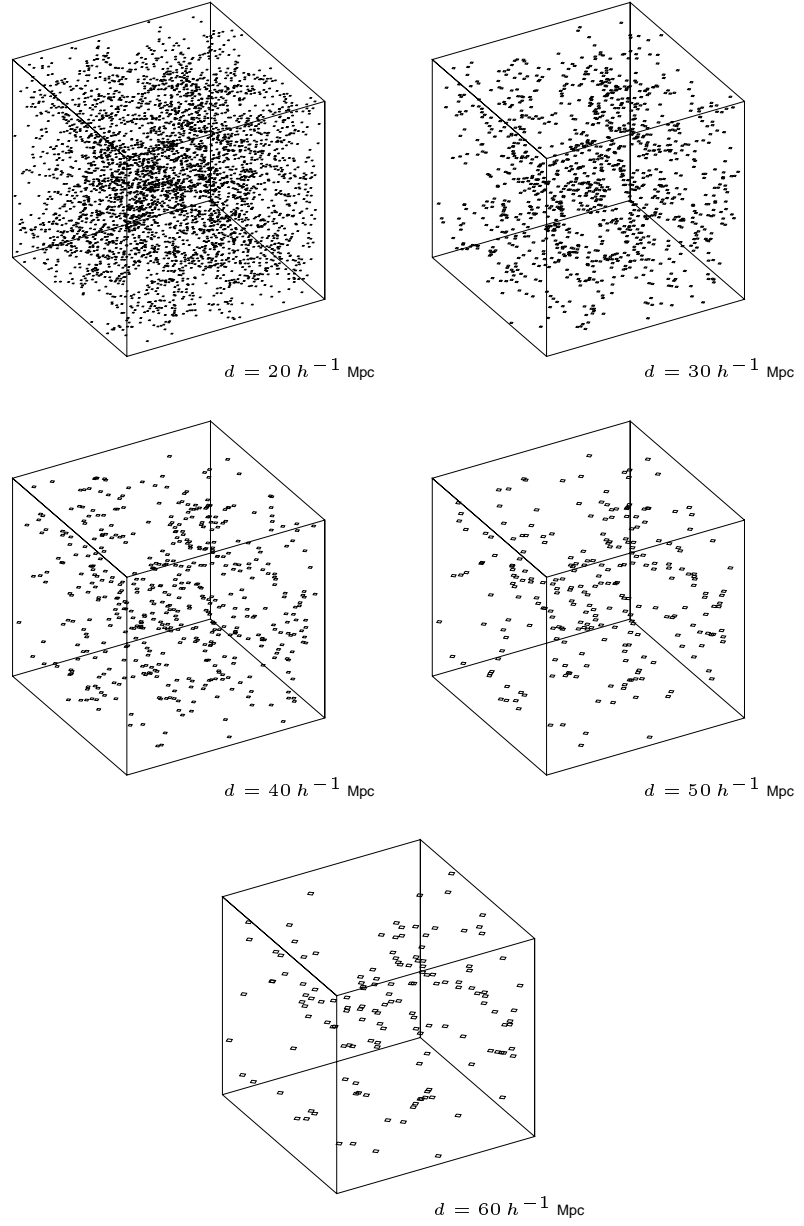


Figure 1. Cluster samples drawn from the simulations in a cubic volume of side  $320 h^{-1} \text{ Mpc}$ . The mean interparticle distance  $d$  is indicated in each case.

lower panel) is observed. In the plots of  $D_2$  as a function of the scale, we can clearly see these two kinds of behaviour. First a noticeable plateau, in which the value of  $D_2$  oscillates without any clear trend around a constant value for each sample. This happens for small scales. For larger scales the local value of  $D_2$  increases with  $r$ , indicating a clear tendency to homogeneity, which is formally reached around  $100 h^{-1} \text{ Mpc}$ , where  $D_2 \sim 3$ .

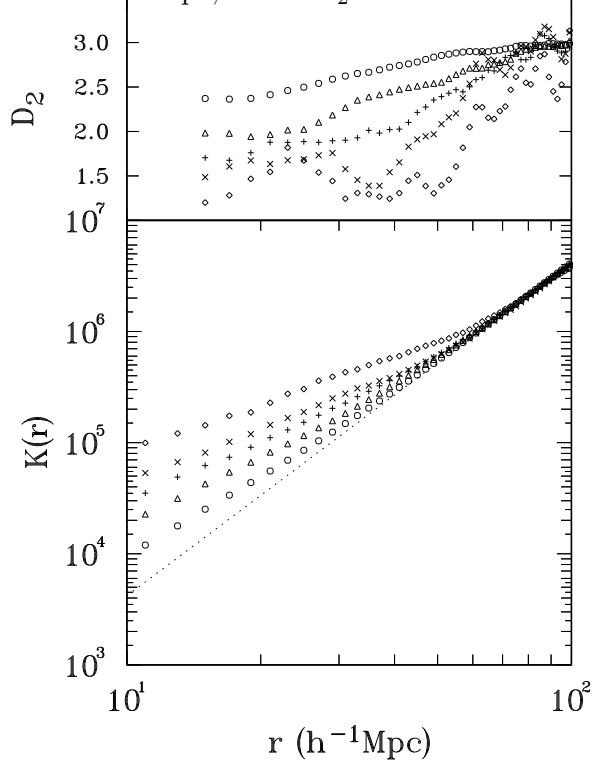


Figure 2. In the bottom panel we see the  $K$ -function for the cluster samples shown in Figure 1. The dotted line corresponds to the expected theoretical function of a Poisson distribution (slope = 3). The different point marks correspond from bottom to top to progressively higher values of the mean interparticle separation. The top panel shows the variation of the local slope  $D_2$  with the scale.

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